Comparing the Lee–Yang gap with the mass gap in the sub-critical planar Ising model

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Abstract

We prove that in the 2D classical Ising model, the subcritical mass gap is comparable to the Lee–Yang gap raised to the power 8/15. Ursell-function monotonicity and a tree bound give the upper bound; a self-contained proof of the monotonicity is given in the appendix. The lower bound follows from the Lee–Yang product, the susceptibility and the critical magnetization. We also obtain uniform factorial bounds on the field derivatives of the pressure.

1 Statement and notation

Our question is motivated by the analogy between classical statistical mechanics and quantum mechanics. In that context, consider the Combes–Thomas estimate of [AW15, Section 10.3] which relates distance from the spectrum to spatial decay of the Green's function: for a Schrödinger operator $H = -\Delta + V(X)$ on $\ell^2(\mathbb{Z}^d)$, if $0 < \delta = \operatorname{dist}(E, \operatorname{spec} H) \leq 1$, then $|(H - E\mathbb{1})^{-1}(x, y)| \leq C\delta^{-1}e^{-c\delta|x-y|_1}$ for all $x, y \in \mathbb{Z}^d$. For real energies below the spectrum, $E < \inf \sigma(H)$, the estimate improves to

$$|(H - E\mathbb{1})^{-1}(x, y)| \leq C\delta^{-1}e^{-c\sqrt{\delta}|x-y|_1} \qquad (x, y \in \mathbb{Z}^d). \tag{1.1}$$

Thus, writing $m_H(E)$ for the optimal uniform exponential decay rate of the Green's function, we obtain

$$m_H(E) \gtrsim \sqrt{\delta}\,. \tag{1.2}$$

Of course we always have via Feynman–Kac

$$m_H(E) \leq \sqrt{\delta + \sup V - \inf \sigma(H)}\,. \tag{1.3}$$

It is tempting to ask whether an analogous relation may hold for the Ising model in its sub-critical, disordered phase, because one may associate the Green's function with the two-point function and the distance to the spectrum with *the Lee-Yang gap*: the magnitude of the closest Lee-Yang zero to the origin. The association of these two quantities is not outlandish by itself. Penrose and Lebowitz [PL74] prove that correlations decay exponentially at positive real fugacities in a connected zero-free region containing zero, provided this region is uniform for sufficiently large periodic rectangles. At the fugacity corresponding to zero magnetic field, a strictly positive Lee-Yang gap supplies their zero-free hypothesis, while exponential decay gives a positive mass gap. Their result thus provides a qualitative precedent, but not a quantitative comparison of the two gaps. Furthermore, in one dimension, where the critical point is at zero temperature, $\beta = \infty$, Lee and Yang's calculation [LY52, Section V.E, formulas (43)–(44)] places the Lee–Yang gap at $H = \mathrm{i}\arcsin(e^{-2K})$, with K the dimensionless coupling and H the dimensionless field. Here, as below, the field enters as βh, so $K = \beta$ and $H = \beta h$. Combining this with the routine zero-field transfer-matrix calculation gives

$$m_\beta^{(1\mathrm{D})} = -\log\tanh\beta, \qquad \mathcal{L}_\beta^{(1\mathrm{D})} = \frac{1}{\beta}\arcsin(e^{-2\beta});$$

Consequently,

$$m_\beta^{(1D)} \sim 2e^{-2\beta}, \qquad \mathcal{L}_\beta^{(1D)} \sim \frac{e^{-2\beta}}{\beta}, \qquad \Longrightarrow \quad m_\beta^{(1D)} \sim 2\beta\mathcal{L}_\beta^{(1D)} \quad (\beta \to \infty).$$

Thus the two gaps are asymptotically proportional when the Lee–Yang gap is measured in the dimensionless field βh. In this note we compare these two gaps in the planar, $\mathbb{Z}^2$, Ising model as $\beta \uparrow \beta_c = \frac{1}{2}\log(1+\sqrt{2}) < \infty$.

For a nonempty finite $\Lambda \subset \mathbb{Z}^2$, let $E(\Lambda)$ be its nearest-neighbour edges and set

$$Z_{\beta,\Lambda}(h) = \sum_{\sigma\in\{-1,1\}^\Lambda} \exp\left(\beta \sum_{\{x,y\}\in E(\Lambda)} \sigma_x\sigma_y + \beta h M_\Lambda\right), \qquad M_\Lambda = \sum_{x\in\Lambda}\sigma_x. \tag{1.4}$$

All finite domains have free boundary conditions. Write $\mathbb{E}_{\beta,\Lambda,h}$ for the corresponding expectation, omitting h at zero field and omitting Λ when $\Lambda = \mathbb{Z}^2$. For $L \in \{0,1,2,\dots\}$, put $\Lambda_L = [-L,L]^2 \cap \mathbb{Z}^2$, $M_L = M_{\Lambda_L}$ and $Z_{\beta,L} = Z_{\beta,\Lambda_L}$. Onsager's critical-point relation [Ons44, formula (121)] is $\sinh(2K_1)\sinh(2K_2) = 1$, where K_1, K_2 are the dimensionless couplings in the two lattice directions. The isotropic choice $K_1 = K_2 = \beta$ gives the critical inverse temperature $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$. For $0 < \beta < \beta_c$, *the axial mass gap* is

$$m_\beta = -\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}_\beta[\sigma_0\sigma_{(n,0)}]. \tag{1.5}$$

The transfer-matrix calculation in [SML64, formula (3.36a)] gives the zero-momentum excitation energy $\epsilon_0 = 2(K_1^* - K_2)$ in the disordered phase, with dual coupling $K_1^* = -\frac{1}{2}\log\tanh K_1$. For $K_1 = K_2 = \beta$, this is the axial decay rate m_β; expanding at β_c gives

$$m_\beta = -2\beta - \log\tanh\beta = 4\,(\beta_c - \beta) + O(|\beta_c - \beta|^2)\,. \tag{1.6}$$

Define, in the physical field convention of (1.4), the *Lee-Yang gap* as

$$\mathcal{L}_{\beta,\Lambda} = \inf\{s > 0 : Z_{\beta,\Lambda}(is) = 0\}, \qquad \mathcal{L}_\beta = \inf_{L\geq 0} \mathcal{L}_{\beta,\Lambda_L}. \tag{1.7}$$

The Lee–Yang theorem [LY52] places the zeros of the ferromagnetic Ising partition polynomial on the fugacity circle $|z| = 1$. Since $z = e^{-2\beta h}$ in the convention of (1.4), this is precisely $\mathrm{Re}\{h\} = 0$, so all zeros lie on the imaginary axis. The results of Penrose and Lebowitz [PL74, Theorems 1–2] give a transfer-matrix gap and exponential decay at positive real fugacities in a connected zero-free region containing zero, provided that region persists for sufficiently large periodic rectangles. At the fugacity corresponding to zero magnetic field, the decay estimate becomes $\mathbb{E}_\beta[\sigma_0\sigma_x] \leq Ce^{-c|x|}$ with $c > 0$, and hence $m_\beta > 0$, under their uniform Lee–Yang-gap hypothesis. There are (to our knowledge) no known rigorous results on $\mathcal{L}_\beta$ analogous to (1.6), only a scaling *prediction*. In the continuum description of Fonseca and Zamolodchikov [FZ03], the Lee–Yang gap occurs at a fixed imaginary value of $h_{\mathrm{cont}}/|m_{\mathrm{cont}}|^{15/8}$, where h_{cont} is the magnetic coupling and m_{cont} the thermal mass parameter. These correspond, to leading order and up to nonzero normalization constants, to βh and $\beta - \beta_c$, respectively. In our field convention, the prediction is therefore

$$\mathcal{L}_\beta \sim A\,(\beta_c - \beta)^{15/8} \tag{1.8}$$

for some $A \in (0,\infty)$.

The following result gives the near-critical power relating the gaps for free square boxes.

Theorem 1.1. *There are $c, C > 0$ and $0 < \beta_0 < \beta_c$ such that,*

$$c\mathcal{L}_\beta^{8/15} \leq m_\beta \leq C\mathcal{L}_\beta^{8/15} \qquad (\beta \in (\beta_0, \beta_c))\,. \tag{1.9}$$

Moreover, for every $L \geq 0$ and $k \geq 1$,

$$\frac{1}{|\Lambda_L|}\left|\partial_h^{2k} \log Z_{\beta,L}(h)\big|_{h=0}\right| \leq (2k)!C^{2k}m_\beta^{2-15k/4}. \tag{1.10}$$

We leave open whether the limit

$$\lim_{\beta\to\beta_c^-} \frac{\mathcal{L}_\beta^{8/15}}{m_\beta} \tag{1.11}$$

exists. By Theorem 1.1, any such limit would necessarily belong to $(0,\infty)$.

Where does 15/8 come from? Wu's critical asymptotic [Wu66] gives the decay $\mathbb{E}_{\beta_c}[\sigma_0\sigma_x] \asymp |x|^{-1/4}$ for the zero-field square-lattice model. This is the input that suggests the exponent 15/8. Indeed, consider the block magnetization $M_R = \sum_{x\in\Lambda_R} \sigma_x$ in the infinite-volume critical state. Since $\mathbb{E}_{\beta_c}[M_R] = 0$,

$$\begin{aligned}\mathbb{V}\mathrm{ar}_{\beta_c}(M_R) = \mathbb{E}_{\beta_c}[M_R^2] &= \sum_{x,y\in\Lambda_R} \mathbb{E}_{\beta_c}[\sigma_x\sigma_y] \\ &\asymp R^2\left(1 + \sum_{r=1}^{R} r\cdot r^{-1/4}\right) \\ &\asymp R^2\left(1 + \int_1^R r^{3/4}\,dr\right) \\ &= R^2\left(1 + \frac{4}{7}(R^{7/4}-1)\right) \asymp R^{15/4}.\end{aligned}$$

Taking the square root gives the fluctuation scale

$$\sqrt{\mathbb{V}\mathrm{ar}_{\beta_c}(M_R)} \asymp R^{15/8}.$$

The field term $\beta h M_R$ becomes significant when $hR^{15/8}$ is of order one. Cutting off at the correlation length $R \asymp m_\beta^{-1}$ predicts $h \asymp m_\beta^{15/8}$. The proof obtains this scale by summing a tree bound for Ursell functions, whose edge weights are square roots of the two-point function.

Higher dimensions. The Ursell-function argument below is valid in every fixed dimension. For the nearest-neighbour Ising model on $\mathbb{Z}^d$, $d \geq 2$, a critical bound $\mathbb{E}_{\beta_c}[\sigma_0\sigma_x] \leq C(1+|x|)^{-a}$, with $0 < a < d$, gives $\chi_\beta := \sum_x \mathbb{E}_\beta[\sigma_0\sigma_x] \leq C m_\beta^{-(d-a)}$ and $Q_\beta := \sum_x \mathbb{E}_\beta[\sigma_0\sigma_x]^{1/2} \leq C m_\beta^{-(d-a/2)}$ by the same spectral argument. The zero-free criterion below then yields $\mathcal{L}_\beta \geq (e\beta Q_\beta)^{-1} \geq c m_\beta^{d-a/2}$. In the customary notation $a = d-2+\eta$, the exponent is $(d+2-\eta)/2$, recovering 15/8 when $d = 2$. A matching upper bound on $\mathcal{L}_\beta$ additionally requires a near-critical susceptibility lower bound $\chi_\beta \geq c m_\beta^{-(d-a)}$ and a critical magnetization upper bound $\mathbb{E}_{\beta_c,h}[\sigma_0] \leq C h^{a/(2d-a)}$.

What about RFIM? For random fields, spatial decay and a complex-field gap must be distinguished. Given real fields $\eta = (\eta_x)$, replace $\beta h M_\Lambda$ in (1.4) by $\beta\sum_{x\in\Lambda}(h+\eta_x)\sigma_x$, and denote the resulting partition function by $Z^\eta_{\beta,\Lambda}(h)$. Set

$$\mathcal{L}^\eta_{\beta,\Lambda} = \inf\{|h| : Z^\eta_{\beta,\Lambda}(h) = 0\}, \qquad \mathcal{L}^\eta_\beta = \inf_{L\geq 0} \mathcal{L}^\eta_{\beta,\Lambda_L}.$$

These zeros need not lie on the imaginary axis. We conjecture that the near-critical lower bound remains uniform in the background field.

Conjecture 1.2. *There are $c > 0$ and $\beta_0 < \beta_c$ such that, for $\beta_0 < \beta < \beta_c$ and every real field configuration η,*

$$\mathcal{L}^\eta_\beta \geq c m_\beta^{15/8}.$$

Here m_β is the mass of the zero-field model.

Writing $\mathbf{E}_\eta$ for disorder expectation, Conjecture 1.2 would imply $\mathbf{E}_\eta[\mathcal{L}^\eta_\beta] \geq c m_\beta^{15/8}$ for any field law. A uniform reverse inequality is impossible: for $\eta_x = a \neq 0$, one has $Z^\eta_{\beta,\Lambda}(h) = Z_{\beta,\Lambda}(h+a)$ and hence $\mathcal{L}^\eta_\beta = \sqrt{a^2 + \mathcal{L}^2_\beta} \geq |a|$, whereas $m_\beta \to 0$. Averaging the partition function gives a different object: for independent identically distributed symmetric fields with $\mathbf{E}_\eta e^{\beta|\eta_0|} < \infty$, independence yields $\mathbf{E}_\eta Z^\eta_{\beta,\Lambda}(h) = (\mathbf{E}_\eta e^{\beta\eta_0})^{|\Lambda|} Z_{\beta,\Lambda}(h)$, so its zeros are exactly those of the pure model.

For the RFIM itself, we expect a separation of the two gaps below the critical temperature of the pure model. Let the η_x be independent centered Gaussian variables of variance $\varepsilon^2 > 0$, write $\mathbb{E}^\eta_\beta$ for infinite-volume thermal expectation at $h = 0$, and define the mass from the disorder-averaged connected correlation by

$$m^{\mathrm{RF}}_{\beta,\varepsilon} := \liminf_{n\to\infty} -\frac{1}{n}\log \mathbf{E}_\eta\left[\mathbb{E}^\eta_\beta[\sigma_0\sigma_{(n,0)}] - \mathbb{E}^\eta_\beta[\sigma_0]\mathbb{E}^\eta_\beta[\sigma_{(n,0)}]\right].$$

Ding and Xia [DX21, Theorem 1.1] establish the averaged boundary-influence estimate $\mathbf{E}[\langle\sigma_0\rangle_{\Lambda_N,+} - \langle\sigma_0\rangle_{\Lambda_N,-}] \le Ce^{-cN}$ for every finite $\beta > 0$ and independent centered Gaussian fields of variance $\varepsilon^2 > 0$. Here $+$ and $-$ denote boundary conditions, and the constants may depend on β, ε. For our field η, taking $N = \lfloor n/2 \rfloor$ bounds the disorder-averaged connected correlation between 0 and $(n, 0)$ and yields $m^{\mathrm{RF}}_{\beta,\varepsilon} > 0$. We conjecture that nevertheless

$$\mathbf{E}_\eta[\mathcal{L}^\eta_\beta] = 0 \qquad \text{for every } \beta > \beta_c \text{ and } \varepsilon > 0,$$

which, by nonnegativity, is equivalent to $\mathcal{L}^\eta_\beta = 0$ almost surely. The heuristic is that arbitrarily large regions with nearly vanishing random field resemble the ordered pure model and can bring zeros close to the origin; controlling their effect on the full partition function remains part of the conjecture. The localization results in [AW15, Sections 6.3 and 7.1] exhibit the analogous phenomenon for random Schrödinger operators: in the localization regime, an eigenfunction may satisfy $H\psi_E = E\psi_E$ and $|\psi_E(x)| \le C_{\psi_E} e^{-c|x-x_E|}$, with $E \in \sigma(H)$ and localization centre x_E. Thus spatial decay can coexist with zero distance from the spectrum, paralleling the proposed coexistence of a positive mass and a closed Lee–Yang gap.

The proof of Theorem 1.1 treats the upper and lower bounds separately. The lower bound follows readily from the Lee–Yang product and standard planar Ising estimates, and is presented in Section 3. For the upper bound, Section 4 uses the signs and coupling monotonicity of even Ursell functions to obtain a tree bound and a zero-free criterion. In Section 4, near-critical two-point estimates verify this criterion on the scale $m_\beta^{15/8}$; the Lee–Yang product then yields the bounds on field derivatives. The required Ursell-function signs and monotonicity are proved in Section A. An alternative proof by a finite block expansion and planar crossing bounds is given in an earlier version of this manuscript but omitted here for brevity. In Section A.1, we explain the issue in the constrained auxiliary inequality used in [CJN23]; the component proof in Section A avoids this inequality.

The exponent 15/8 is consistent with the continuum scaling description of the planar Ising model. Fonseca and Zamolodchikov [FZ03] study the analytic structure of the free-energy scaling function in the variable $h_{\mathrm{cont}}/|m_{\mathrm{cont}}|^{15/8}$, where h_{cont} is the continuum magnetic coupling and m_{cont} the thermal mass parameter, and investigate the Yang–Lee edge through numerical calculations and analyticity arguments. The leading-order identifications $h_{\mathrm{cont}} \propto \beta h$ and $|m_{\mathrm{cont}}| \propto \beta_c - \beta$ give the same exponent 15/8 in the present comparison. For the lattice model, Ott's analyticity theorem [Ott20] states that exponential weak mixing at (β, H) implies analyticity of the pressure in a neighbourhood of that point, with H the dimensionless magnetic field. At $H = 0$ and $\beta < \beta_c$, the required mixing holds; the change of variables $H = \beta h$ therefore gives analyticity near every such $(\beta, 0)$ in our convention. We are not aware of the near-critical comparison $\mathcal{L}_\beta \asymp m_\beta^{15/8}$ appearing earlier in the literature.

Constants c, C may change between occurrences, but are independent of the volume, order and β sufficiently close to β_c.

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2 Preliminaries

Write all positive imaginary zeros as $i\mathcal{L}_{\beta,\Lambda,j}$, counted with multiplicity, with $\mathcal{L}_{\beta,\Lambda,1} = \mathcal{L}_{\beta,\Lambda}$. Hadamard factorization [SS03, Chapter 5, Theorem 5.1] of the even entire function $Z_{\beta,\Lambda}$, of order one, gives the paired Lee–Yang product

$$\mathbb{E}_{\beta,\Lambda}\left[\exp\left(\beta h M_\Lambda\right)\right] = \frac{Z_{\beta,\Lambda}(h)}{Z_{\beta,\Lambda}(0)} = \prod_{j\ge 1}\left(1 + \frac{h^2}{\mathcal{L}^2_{\beta,\Lambda,j}}\right). \tag{2.1}$$

Hence

$$\partial_h \log\left(\mathbb{E}_{\beta,\Lambda}\left[\exp\left(\beta h M_\Lambda\right)\right]\right) = \beta \mathbb{E}_{\beta,\Lambda,h}\left[M_\Lambda\right] = \sum_{j\ge 1} \frac{2h}{\mathcal{L}^2_{\beta,\Lambda,j} + h^2}.$$

From this we obtain

$$\beta \mathbb{E}_{\beta,\Lambda,h}[M_\Lambda] \geq h \sum_{j\geq 1} \frac{1}{\mathcal{L}^2_{\beta,\Lambda,j}} = h\frac{\beta^2}{2}\,\mathbb{V}\mathrm{ar}_{\beta,\Lambda}[M_\Lambda] \qquad (0 \leq h \leq \mathcal{L}_{\beta,\Lambda})\,. \tag{2.2}$$

Notation Throughout, we write

1. $M_\Lambda := \sum_{x\in\Lambda} \sigma_x$ for the magnetization in Λ.
2. $\chi_\beta \equiv \sum_{x\in\mathbb{Z}^2} \mathbb{E}_\beta[\sigma_0\sigma_x]$ for the infinite-volume susceptibility.

3 The lower bound

We begin with a short

Lemma 3.1. *For $\beta < \beta_c$ sufficiently close to β_c and all $h > 0$,*

$$\chi_\beta \geq c m_\beta^{-7/4}, \qquad \mathbb{E}_{\beta_c,h}[\sigma_0] \leq C h^{1/15}. \tag{3.1}$$

Proof. Write $t = \beta_c - \beta$. The susceptibility asymptotic in [Wu+76, equations (2.43)–(2.45)] is

$$T\chi(T) \sim C_{0,+}(1 - T_c/T)^{-7/4} \qquad (T \downarrow T_c),$$

where T is the temperature, $\chi(T)$ is the zero-field susceptibility with respect to the physical magnetic field, and $C_{0,+} > 0$. Taking their couplings $E_1 = E_2 = 1$ and $T = \beta^{-1}$, we have $T_c = \beta_c^{-1}$ and $1 - T_c/T = t/\beta_c$. Their susceptibility therefore gives

$$\chi_\beta = T\chi(T) \sim C_{0,+}\beta_c^{7/4} t^{-7/4}.$$

Since $m_\beta \sim 4t$ by (1.6), this proves the first estimate.

For the second estimate, [CGN14, Theorem 1.1] bounds the infinite-volume critical magnetization by $CH^{1/15}$ for small $H > 0$, where H is the coefficient of $\sum_x \sigma_x$ in the exponent of the Gibbs weight. Setting $H = \beta_c h$ gives $\mathbb{E}_{\beta_c,h}[\sigma_0] \leq Ch^{1/15}$ for small $h > 0$, after adjusting C. Since the magnetization is at most one, increasing C extends this bound to all $h > 0$. □

With this, we are now ready for the

Proof of the lower bound in (1.9). We may assume $\mathcal{L}_\beta > 0$, since otherwise the claimed bound is immediate. Griffiths' inequalities and the summable bound (4.13) give

$$\lim_{L\to\infty} \frac{\mathbb{V}\mathrm{ar}_{\beta,\Lambda_L}[M_L]}{|\Lambda_L|} = \chi_\beta. \tag{3.2}$$

For the lower limit, restrict to bounded displacements and sites far from the boundary before letting these cutoffs grow; the reverse bound follows from infinite-volume domination. For $0 < h \leq \mathcal{L}_\beta$, (2.2), gives

$$\frac{\beta h}{2}\,\mathbb{V}\mathrm{ar}_{\beta,\Lambda_L}[M_L] \leq \mathbb{E}_{\beta,\Lambda_L,h}[M_L] \leq |\Lambda_L|\mathbb{E}_{\beta_c,h}[\sigma_0], \tag{3.3}$$

where the last inequality is monotonicity. Divide by $|\Lambda_L|$, let $L \to \infty$, and use Lemma 3.1 at $h = \mathcal{L}_\beta > 0$:

$$c\mathcal{L}_\beta m_\beta^{-7/4} \leq \frac{\beta\mathcal{L}_\beta}{2}\chi_\beta \leq \mathbb{E}_{\beta_c,\mathcal{L}_\beta}[\sigma_0] \leq C\mathcal{L}_\beta^{1/15}. \tag{3.4}$$

Hence $c\mathcal{L}_\beta^{8/15} \leq m_\beta$, as required. □

4 The upper bound

We now prove the upper bound $m_\beta \leq C\mathcal{L}_\beta^{8/15}$ in (1.9). Equivalently, we must show that $\mathcal{L}_\beta \geq c m_\beta^{15/8}$, i.e., every finite free square must have no partition-function zeros in a disk of radius $\asymp m_\beta^{15/8}$ about the origin. We first obtain a zero-free criterion in terms of a sum of square roots of two-point functions, and then estimate this sum using the mass gap.

The link between correlations and zeros is the Taylor expansion of $\log Z_{\beta,\Lambda}$ at zero field. Its coefficients are sums of Ursell functions, which measure joint spin cumulants. A tree bound reduces these many-point quantities to products of two-point functions. Summing the trees will give an explicit lower bound on the radius of convergence of the Taylor series for $\log Z_{\beta,\Lambda}$, and hence a disk in which $Z_{\beta,\Lambda}$ cannot vanish.

4.1 Taylor coefficients and Ursell functions

Fix a nonempty finite $\Lambda \subset \mathbb{Z}^2$. Since $Z_{\beta,\Lambda}(0) > 0$, the ratio $Z_{\beta,\Lambda}(h)/Z_{\beta,\Lambda}(0)$ has an analytic logarithm in some neighbourhood of $h = 0$. Choose the branch that vanishes at zero and write

$$K_{\beta,\Lambda}(h) := \log \frac{Z_{\beta,\Lambda}(h)}{Z_{\beta,\Lambda}(0)} = \log \mathbb{E}_{\beta,\Lambda}[e^{\beta h M_\Lambda}] = \sum_{n\geq 1} \frac{\partial_h^n \log Z_{\beta,\Lambda}(0)}{n!} h^n. \tag{4.1}$$

At this point we only know that the series converges sufficiently close to zero; obtaining a quantitative radius, uniform in Λ, is the purpose of the argument.

For any finite ferromagnetic Ising graph at zero field, let I be a set of labels, corresponding to (possibly repeated) marked sites x_i for each $i \in I$. The joint Ursell function is defined by:

$$u(I) = \left(\prod_{i\in I} \partial_{t_i}\right) \log \mathbb{E}\left[\exp\left(\sum_{i\in I} t_i \sigma_{x_i}\right)\right]\Bigg|_{t=0}. \tag{4.2}$$

Thus $u(I)$ is the joint cumulant of the marked spins. By spin-flip symmetry, all odd cumulants vanish, and at zero field the two-point Ursell function is simply $u_2(\sigma_x, \sigma_y) = \mathbb{E}[\sigma_x\sigma_y] - \mathbb{E}[\sigma_x]\mathbb{E}[\sigma_y]$. Differentiating along the uniform field βh gives the coefficient identity:

$$\partial_h^n \log Z_{\beta,\Lambda}(0) = \beta^n \sum_{(x_1,\dots,x_n)\in\Lambda^n} u_n(\sigma_{x_1}, \dots, \sigma_{x_n}). \tag{4.3}$$

The sum runs over ordered tuples, including repeated sites; this is why all the estimates below must allow repetitions. To obtain a lower bound on the radius of convergence in (4.1), we therefore need estimates on the sums in (4.3) for all even orders (the odd orders vanish by spin-flip symmetry). More precisely, a bound of the form $A_\Lambda B^n$ on the absolute value of the nth Taylor coefficient, with B independent of Λ, gives convergence for $|h| < B^{-1}$. The identity theorem then extends the local identity

$$\frac{Z_{\beta,\Lambda}(h)}{Z_{\beta,\Lambda}(0)} = \exp K_{\beta,\Lambda}(h)$$

to this whole disk. Since the exponential never vanishes, $\mathcal{L}_{\beta,\Lambda} \geq B^{-1}$, and taking the infimum over finite free squares gives $\mathcal{L}_\beta \geq B^{-1}$. The next two steps identify such a growth rate in terms of two-point functions.

4.2 From coupling monotonicity to a tree bound

We first work on an arbitrary finite ferromagnetic Ising graph, with couplings $J_{uv} \geq 0$ and zero field. We shall use Theorem A.1 below. It yields, for $|I|$ even,

$$(-1)^{|I|/2-1} u(I) \geq 0 \qquad (\text{“Sign rule”}) \tag{4.4}$$

as in [Shl86], and the coupling monotonicity

$$(-1)^{k-1} \partial_{J_{uv}} u_{2k}(\sigma_{x_1}, \dots, \sigma_{x_{2k}}) \geq 0 \qquad (k \geq 1). \tag{4.5}$$

Their role is to bound a cumulant of a given order by products of cumulants of lower orders. To see this, take an even label set I with $|I| \geq 4$, and choose distinct labels $a, b \in I$. Suppose first that $x_a \neq x_b$; if necessary, add the edge $\{x_a, x_b\}$ with coupling zero. Differentiating with respect to this coupling inserts $\sigma_{x_a}\sigma_{x_b}$ and applying the product rule gives:

$$\partial_{J_{x_a x_b}} u(I \setminus \{a,b\}) = u(I) + \sum_{\substack{B\subset I:\ a\in B,\ b\notin B \\ |B| \text{ even}}} u(B)u(I\setminus B). \tag{4.6}$$

For $|I| = 2k$, multiply (4.6) by $(-1)^{k-2}$. The derivative on the left is nonnegative by (4.5), applied at order $2k-2$. The sign rule changes the first term on the right to $-|u(I)|$, and each product to $|u(B)|\,|u(I\setminus B)|$. Rearranging yields

$$|u(I)| \leq \sum_{\substack{B\subset I:\ a\in B,\ b\notin B \\ |B| \text{ even}}} |u(B)|\,|u(I\setminus B)|. \tag{4.7}$$

If instead $x_a = x_b$, we may perturb by $t\sigma_{x_a}^2 = t$ and differentiate so (4.7) holds with equality. The splitting into products in (4.7) is reminiscent of the product structure arising from finite-rank resolvent identities; see [AW15, Section 5.5, pp. 81–83].

The recursive splitting in (4.7) suggests organizing the bound by ways of connecting the marked spins through pair correlations. A tree on I consists of vertices, namely the labels $i \in I$, and edges, namely unordered pairs $\{i, j\}$ of distinct labels, with exactly one path between any two vertices. Thus a tree connects all $|I|$ labels using $|I| - 1$ edges and contains no cycles. An edge records a correlation between the marked spins at x_i and x_j, regardless of their distance; distinct labels may even mark the same site. This organization matches the recursive splitting: joining two trees on disjoint label sets by a single edge produces a tree on their union. In the following bound, each edge carries the square root of the corresponding two-point correlation.

We have the following

Lemma 4.1. *Writing $G_{ij} = \mathbb{E}[\sigma_{x_i}\sigma_{x_j}]$ to be the two-point correlations, we know by Griffiths' inequality that $0 \le G_{ij} \le 1$ for all i, j. Define $\mathcal{T}(I)$ to be the set of trees with vertex set I. Then*

$$|u(I)| \le \sqrt{\max_{\substack{i,j\in I\\ i\neq j}} G_{ij}} \sum_{T\in\mathcal{T}(I)} \prod_{\{i,j\}\in E(T)} \sqrt{G_{ij}}. \tag{4.8}$$

Proof. We apply induction on the even number $|I|$. With two labels there is only one tree, and the right side is $G_{12} = |u(I)|$. For the induction step, choose a, b attaining $G_* := \max_{i\neq j} G_{ij}$ and apply (4.7). For each admissible B, apply the inductive tree estimate on B and $I \setminus B$. Each of the two inductive prefactors is at most $\sqrt{G_*}$, so their product is at most

$$G_* = \sqrt{G_*}\sqrt{G_{ab}}.$$

Keep the first factor as the prefactor in (4.8); interpret the second as the weight of the edge $\{a, b\}$. Adding this edge joins the two smaller trees into a tree on I.

For the fixed pair a, b, this construction is injective. Indeed, deleting $\{a, b\}$ from a resulting tree recovers both smaller trees, and the component containing a recovers B. Thus the sum produced by the splitting inequality contains no repeated trees. It is bounded by the sum over all trees on I, since all edge weights are nonnegative. This completes the induction. □

4.3 Summing the trees and obtaining a zero-free disk

We now return to the lattice model and define

$$Q_{\beta,\Lambda} := \sup_{x\in\Lambda} \sum_{y\in\Lambda} \sqrt{\mathbb{E}_{\beta,\Lambda}[\sigma_x\sigma_y]}. \tag{4.9}$$

Fix a tree T on a label set $\{1, \dots, n\}$ and root it at label 1. If $i \neq 1$ is a leaf and j its unique neighbour, then x_i occurs in only one factor of the product of G_{ij} over edges of T. For any fixed choice of the other marked sites x_j, the sum over x_i gives:

$$\sum_{x_i\in\Lambda} \sqrt{\mathbb{E}_{\beta,\Lambda}[\sigma_{x_i}\sigma_{x_j}]} \le Q_{\beta,\Lambda}.$$

Remove that leaf and repeat. The $n - 1$ non-root labels contribute at most one factor $Q_{\beta,\Lambda}$ each, while the final sum over x_1 contributes $|\Lambda|$. Consequently,

$$\sum_{(x_1,\dots,x_n)\in\Lambda^n} \prod_{\{i,j\}\in E(T)} \sqrt{\mathbb{E}_{\beta,\Lambda}[\sigma_{x_i}\sigma_{x_j}]} \le |\Lambda| Q_{\beta,\Lambda}^{n-1}. \tag{4.10}$$

The prefactor in (4.8) is at most one and may now be dropped. Combining the coefficient identity (4.3), the tree bound, and (4.10), and using Cayley's formula $|\mathcal{T}(\{1, \dots, n\})| = n^{n-2}$ for the number of trees on n labelled vertices (see [AZ14, Chapter 32]), gives, for even $n \ge 2$,

$$\frac{|\partial_h^n \log Z_{\beta,\Lambda}(0)|}{n!} \le \frac{\beta^n |\Lambda| n^{n-2} Q_{\beta,\Lambda}^{n-1}}{n!} \le \frac{|\Lambda|}{Q_{\beta,\Lambda}} (e\beta Q_{\beta,\Lambda})^n. \tag{4.11}$$

The last inequality uses $n! \geq (n/e)^n$. We therefore have the uniform bound:

$$S_{\beta,\Lambda} := \sum_{n\geq 1} \frac{|\partial_h^n \log Z_{\beta,\Lambda}(0)|}{n!}|h|^n \leq \frac{|\Lambda|}{Q_{\beta,\Lambda}} \sum_{k\geq 1} (e\beta Q_{\beta,\Lambda}|h|)^{2k} \qquad (|h| < (e\beta Q_{\beta,\Lambda})^{-1}).$$

and hence

$$\mathcal{L}_{\beta,\Lambda} \geq (e\beta Q_{\beta,\Lambda})^{-1}. \tag{4.12}$$

4.4 Two-point estimates at the mass scale

We begin with an upper bound to complement Lemma 3.1.

Lemma 4.2. *Uniformly in finite $\Lambda \subset \mathbb{Z}^2$, $x, y \in \Lambda$ and $\beta < \beta_c$ sufficiently close to β_c,*

$$0 \leq \mathbb{E}_{\beta,\Lambda}[\sigma_x \sigma_y] \leq C(1 + |x-y|_\infty)^{-1/4} e^{-m_\beta |x-y|_\infty/2}. \tag{4.13}$$

Consequently,

$$\sup_{x\in\Lambda} \sum_{y\in\Lambda} \mathbb{E}_{\beta,\Lambda}[\sigma_x \sigma_y] \leq \chi_\beta \leq C m_\beta^{-7/4}. \tag{4.14}$$

Proof. The Griffiths–Kelly–Sherman inequalities [KS68] give nonnegativity of spin-product expectations and monotonicity in each ferromagnetic coupling when the fields are nonnegative. At zero field, adding the missing lattice couplings with strength β thus gives $0 \leq \mathbb{E}_{\beta,\Lambda}[\sigma_x\sigma_y] \leq \mathbb{E}_\beta[\sigma_x\sigma_y]$. Messager–Miracle-Solé monotonicity [MM77; Sch77] compares a general displacement with an axial one: $\mathbb{E}_\beta[\sigma_0\sigma_z] \leq \mathbb{E}_\beta[\sigma_0\sigma_{(n,0)}]$ for $n = |z|_\infty$. With $z = y - x$, translation invariance bounds the finite-volume correlation by this row correlation at $n = |x-y|_\infty$. The transfer-matrix decomposition in [SML64, Section V.B, formulas (5.20)–(5.22)] expresses the row correlation as a sum of nonnegative spectral weights times powers of eigenvalue ratios. For the isotropic choice of both couplings equal to β, its infinite-volume form is $\mathbb{E}_\beta[\sigma_0\sigma_{(j,0)}] = \int_{m_\beta}^\infty e^{-js}\, d\mu_\beta(s)$, where μ_β is a positive measure and s is minus the logarithm of the eigenvalue ratio. Comparing the integrands at $j = n$ and $j = \lfloor n/2 \rfloor$ gives

$$\mathbb{E}_\beta[\sigma_0\sigma_{(n,0)}] \leq e^{-m_\beta(n-\lfloor n/2\rfloor)} \mathbb{E}_\beta[\sigma_0\sigma_{(\lfloor n/2\rfloor,0)}] \leq C e^{-m_\beta n/2}(1+n)^{-1/4}. \tag{4.15}$$

For the last inequality, Wu's critical row asymptotic [Wu66] gives $\mathbb{E}_{\beta_c}[\sigma_0\sigma_{(j,0)}] \sim a_0 j^{-1/4}$ as $j \to \infty$, with $a_0 > 0$. Apply this at $j = \lfloor n/2 \rfloor$ and use monotonicity in β, increasing the constant to cover bounded j. Summation over square shells proves (4.14). □

We can now complete the proof of the upper bound in (1.9) and the derivative bounds in (1.10).

Proof of the upper and derivative bounds. Summing the square root of (4.13) over square shells gives, uniformly in finite Λ,

$$Q_{\beta,\Lambda} \leq C \sum_{r\geq 0} (1+r)^{7/8} e^{-m_\beta r/4} \leq C m_\beta^{-15/8}. \tag{4.16}$$

Then (4.12) gives, uniformly in finite free squares,

$$\mathcal{L}_{\beta,\Lambda} \geq (e\beta Q_{\beta,\Lambda})^{-1} \geq c m_\beta^{15/8}. \tag{4.17}$$

Taking the infimum over Λ_L proves the upper bound in (1.9).

Next, by (2.1), for every $k \geq 1$,

$$\frac{|\partial_h^{2k} \log Z_{\beta,\Lambda}(0)|}{(2k-1)!} = 2 \sum_{j\geq 1} \mathcal{L}_{\beta,\Lambda,j}^{-2k} \leq \mathcal{L}_{\beta,\Lambda}^{2-2k} \partial_h^2 \log Z_{\beta,\Lambda}(0). \tag{4.18}$$

Since $\partial_h^2 \log Z_{\beta,\Lambda}(0) = \beta^2\, \mathbb{V}\mathrm{ar}_{\beta,\Lambda}(M_\Lambda) \leq C|\Lambda| m_\beta^{-7/4}$ by (4.14), the finite-volume gap bound yields (1.10): the power of m_β is $-7/4 + (15/8)(2-2k) = 2 - 15k/4$. □

A A component proof of Ursell-function monotonicity

Correlation inequalities have long played a central role in the study of the Ising model, beginning with the classical inequalities of Griffiths, Kelly–Sherman, Messager–Miracle-Solé, and Lebowitz [KS68; MM77; Leb74]. In this section, we prove a sign rule and a coupling monotonicity inequality for the connected correlations, or Ursell functions, of the ferromagnetic Ising model at zero external field.

Let $G = (V, E)$ be a finite loopless graph, with ferromagnetic couplings $J : E \to \mathbb{R}_{\geq 0}$ and external field $h : V \to \mathbb{C}$. We use the uniformly normalized partition function

$$Z_{G,J}(h) = Z_G(h) := 2^{-|V|} \sum_{\sigma \in \{-1,1\}^V} \exp\left(\sum_{e=\{x,y\} \in E} J_e \sigma_x \sigma_y + \sum_{x \in V} h_x \sigma_x \right). \tag{A.1}$$

For real valued h, we write $\mathbb{P}_{G,J,h}(\sigma)$ and $\langle \cdot \rangle_{G,J,h}$ for the corresponding Gibbs measure and expectation. We suppress G, J, and h when they are clear from the context unless otherwise stated. For any $B \subseteq V$ we write $\sigma_B := \prod_{i \in B} \sigma_i$.

For a tuple of vertices $I = (x_1, \ldots, x_k) \in V^k$, with repetitions allowed, the Ursell function is the joint cumulant of the corresponding spins:

$$u_{k,G,J}(I) := \frac{\partial^k}{\partial h_{x_1} \cdots \partial h_{x_k}} \log Z_{G,J}(h) \bigg|_{h=0}.$$

Equivalently, the moment–cumulant formula gives the connected-correlation expansion [Spe83]

$$u_{k,G,J}(I) = \sum_{\pi \in \mathcal{P}([k])} (-1)^{|\pi|-1} (|\pi| - 1)! \prod_{B \in \pi} \left\langle \prod_{i \in B} \sigma_{x_i} \right\rangle_{G,J,0}. \tag{A.2}$$

where $\mathcal{P}([k])$ denotes the set of partitions of $[k] = \{1, \ldots, k\}$ into nonempty blocks, and $|\pi|$ is the number of blocks of π. At zero external field, the Gibbs measure is invariant under the global spin flip $\sigma \mapsto -\sigma$, so every odd spin moment vanishes. If k is odd, every partition of $[k]$ contains a block of odd size; the connected-correlation expansion therefore gives $u_{k,G,J}(I) = 0$. It thus suffices to consider Ursell functions of even order. The following theorem is the main result of this section.

Theorem A.1. *Let $G = (V, E)$ be a finite loopless graph and let $J : E \to \mathbb{R}_{\geq 0}$. For every integer $k \geq 1$ and every tuple $I = (x_1, \ldots, x_{2k}) \in V^{2k}$, the corresponding Ursell function satisfies the sign rule*

$$(-1)^{k-1} u_{2k}(I) \geq 0. \tag{A.3}$$

Moreover, for every edge $e \in E$,

$$(-1)^{k-1} \frac{\partial}{\partial J_e} u_{2k}(I) \geq 0. \tag{A.4}$$

The sign rule was first proved by Shlosman in 1986 [Shl86]. A proof of coupling monotonicity was presented by Camia, Jiang, and Newman in 2023 [CJN23]. In seeking to formalize their argument, however, we found a counterexample to an intermediate lemma. We describe this counterexample at the end of the section. See Section A.1.

Our proof begins with the covariance formula for the derivative of a spin correlation with respect to an edge coupling. We rewrite this covariance as an expectation over two independent copies of the Ising model represented by agreement/disagreement variables at each vertex. We then apply Aizenman's random-current expansion to the resulting components [Aiz82]. The component structure of the associated multigraph yields a factorization in which Ursell functions on strictly smaller subgraphs appear. This reduction allows us to prove the sign rule and coupling monotonicity simultaneously by induction, first on the order of the Ursell function and, at each fixed order, on the number of vertices of G.

The two-copy identity. We use the duplicate-variable transformation attributed to Percus by Lebowitz [Leb74]. For real h, let σ and τ be independent samples from $\mathbb{P}_{G,J,h}$, and write $\langle \cdot \rangle_{G,J,h}^{\otimes 2}$ for expectation under their joint law. Define the agreement (s) and disagreement (t) variables at each vertex $x \in V$ by

$$s_x = \frac{\sigma_x + \tau_x}{2}, \qquad t_x = \frac{\sigma_x - \tau_x}{2}.$$

Setting $\mathcal{A} = \{(1,0),(-1,0),(0,1),(0,-1)\}$, the map $(\sigma,\tau) \mapsto (s,t)$ is a bijection onto $\mathcal{A}^V$ with inverse given by $(\sigma,\tau) = (s+t, s-t)$. We write the components of the joint Hamiltonian in this representation:

$$L_G(s;h) := 2 \sum_{e=\{x,y\}\in E} J_e s_x s_y + 2 \sum_{x\in V} h_x s_x, \qquad L_G(t) := 2 \sum_{e=\{x,y\}\in E} J_e t_x t_y.$$

The probability of a configuration $(s,t) \in \mathcal{A}^V$ according to the joint law is $2^{-2|V|} Z_G(h)^{-2} e^{L_G(s;h)+L_G(t)}$. Note the external field acts only on the agreement variables $\{s_x\}_{x\in V}$.

Write $m_{a,G}(h) := \langle \sigma_a \rangle_{G,J,h} = \partial_{h_a} \log Z_G(h)$ for the magnetization at $a \in V$. Differentiating with respect to the coupling of an edge $\{u,v\}$ and expressing the resulting covariance using the two independent copies gives

$$\begin{aligned} \frac{\partial}{\partial J_{uv}} m_{a,G}(h) &= \mathrm{Cov}_{G,J,h}(\sigma_a, \sigma_u\sigma_v) \\ &= \frac{1}{2} \left\langle (\sigma_a - \tau_a)(\sigma_u\sigma_v - \tau_u\tau_v) \right\rangle_{G,J,h}^{\otimes 2} \\ &= 2 \left\langle t_a(t_u s_v + s_u t_v) \right\rangle_{G,J,h}^{\otimes 2} \\ &= \frac{2\, 2^{-2|V|}}{Z_G(h)^2} \sum_{(s,t)\in\mathcal{A}^V} t_a(t_u s_v + s_u t_v) e^{L_G(s;h)+L_G(t)}. \end{aligned} \tag{A.5}$$

This identity is the starting point for the current expansion below. It extends analytically to complex h in a neighborhood of the origin, where $Z_G(h) \neq 0$.

Current representation. We use the current representation developed by Aizenman [Aiz82]. A current on G is a vector $n \in \mathbb{N}_0^E$, viewed as a multigraph in which each edge e is replaced by n_e parallel edges. Its degree at x is $d_n(x) := \sum_{e\ni x} n_e$, and its boundary ∂n is the set of vertices of odd degree. We say $x, y \in V$ are *connected* in n if they are connected in the corresponding multigraph.

Expanding the t-interaction in the Hamiltonian and collecting powers by vertex gives

$$e^{L_G(t)} = \sum_{n\in\mathbb{N}_0^E} w_{G,J}(n) \prod_{x\in V} t_x^{d_n(x)}, \qquad w_{G,J}(n) := \prod_{e\in E} \frac{(2J_e)^{n_e}}{n_e!}.$$

Substituting into the two-copy identity (A.5) yields

$$\partial_{J_{uv}} m_{a,G}(h) = \frac{2\, 2^{-2|V|}}{Z_G(h)^2} \sum_{(p,q)} \sum_{n\in\mathbb{N}_0^E} w_{G,J}(n) \sum_{(s,t)\in\mathcal{A}^V} s_q e^{L_G(s;h)} \prod_{x\in V} t_x^{r_n(x)}, \tag{A.6}$$

where (p,q) runs over (u,v) and (v,u), and $r_n(x) := d_n(x) + \mathbf{1}_{\{x=a\}} + \mathbf{1}_{\{x=p\}}$.

If $r_n(x)$ is odd at any vertex x, the two choices $t_x = \pm 1$ cancel, while $t_x = 0$ contributes nothing. Thus only currents with $\partial n = \{a\} \triangle \{p\}$ survive, where Δ denotes the symmetric difference. Fix such a current, let C be the vertex set of the connected component of a in the current, and write $H := G \setminus C$ for the induced graph on $V \setminus C$. We make the following observations with a goal of factorizing the current sum in C and H.

1. We have $p \in C$, since a connected component cannot contain exactly one odd-degree vertex.
2. Every $x \in C$ has $r_n(x) > 0$, so a nonzero contribution requires $t_x = \pm 1$ and $s_x = 0$. Therefore, all s-interactions touching C, and all field terms on C must vanish.
3. The insertion s_q forces $q \notin C$.
4. No current edge crosses from C to its complement, so the current weight factors over C and H.

On C, summing the two possible t-variables at each vertex gives $2^{|C|}$. The sum on H reconstructs the two-copy model with insertion s_q: the empty-boundary condition imposes no additional restriction, since currents with nonempty boundary vanish. As $\langle s_q \rangle_{H,J,h}^{\otimes 2} = m_{q,H}(h)$, this sum contributes $Z_H(h)^2 m_{q,H}(h)$, together with the normalization factor $2^{2|V\setminus C|}$. Combining this with $2^{|C|}$ from the component C and $2^{-2|V|}$ from the original two-copy measure gives a factor of $2^{-|C|}$.

To express this factorization compactly, we define the coefficients for $C \subset V$ and $H = G \setminus C$:

$$B_{a,p}(C) := 2^{-|C|} \sum_{\substack{n \in \mathbb{N}_0^{E(C)} \\ n \text{ connects } C \\ \partial n = \{a\} \triangle \{p\}}} w_{C,J}(n)\,, \tag{A.7}$$

where $E(C)$ consists of edges with both endpoints in C. When $a = p$, the singleton $C = \{a\}$ with zero current is included. Grouping the currents by C and applying the described factorization gives the magnetization identity:

$$\partial_{J_{uv}} m_{a,G}(h) = 2 \sum_{(p,q)} \sum_{\substack{C \subseteq V \\ a,p \in C,\ q \notin C}} B_{a,p}(C)\, \frac{Z_H(h)^2}{Z_G(h)^2} m_{q,H}(h)\,. \tag{A.8}$$

We now introduce the field-dependent pressure

$$P_G(h) := \log Z_G(h) - \log Z_G(0).$$

Using the fact that $m_{a,G}(h) = \partial_{h_a} P_G(h)$, the preceding identity becomes:

$$\partial_{J_{uv}} \partial_{h_a} P_G(h) = 2 \sum_{(p,q)} \sum_{\substack{C \subseteq V \\ a,p \in C,\ q \notin C}} \underbrace{B_{a,p}(C)\, \frac{Z_H(0)^2}{Z_G(0)^2}}_{\geq 0}\, e^{2[P_H(h) - P_G(h)]}\, \partial_{h_q} P_H(h). \tag{A.9}$$

The identity holds for complex fields h near the origin such that $Z_G(h) \neq 0$. Its essential feature is that the magnetization factor belongs to a strictly smaller graph. This is the reduction used in the induction below.

Induction. We prove the sign rule and coupling monotonicity together. For a tuple I of length $2k$,

$$U_{G,J}(I) := (-1)^{k-1} u_{2k,G,J}(I).$$

The two assertions in Theorem A.1 are then $U_G(I) \geq 0$ and $\partial_{J_{uv}} U_G(I) \geq 0$. For an induced subgraph H, we set $U_H(I) = 0$ if any marked vertex in I lies outside H.

Equation (A.9) suggests a double induction: the magnetization factor lives on a smaller graph, while differentiating the exponential produces correlations of lower order. We therefore induct first on the order k, then for each fixed order on the number of vertices. At each step we prove monotonicity first, then obtain the sign rule by increasing the couplings from zero.

The starting point is the independent-spin model. When couplings $J = 0$, we have $P_G(h) = \sum_{x \in V} \log\cosh h_x$, so every cumulant involving more than one vertex vanishes. If all $2k$ marks coincide, then

$$U_{G,0}(I) = (-1)^{k-1}\, \frac{d^{2k}}{dh^{2k}} \log\cosh h \Big|_{h=0} = \frac{d^{2k}}{dz^{2k}} [-\log\cos z] \Big|_{z=0} > 0. \tag{A.10}$$

The last inequality follows because $(-\log\cos z)' = \tan z$ has strictly positive coefficients at every odd degree. Thus $U_{G,0}(I) \geq 0$ at every order. In particular, the theorem holds on a one-vertex graph, where there are no edge couplings to differentiate.

Fix $k \geq 1$ and a graph G with $N \geq 2$ vertices. Assume that both assertions hold at every even order below $2k$ on all finite graphs, and at order $2k$ on all graphs with fewer than N vertices. We will prove both assertions at order $2k$ on G. Fix $I = (a, x_2, \ldots, x_{2k})$ and an edge $\{u, v\}$. Applying the remaining $2k - 1$ field derivatives to (A.9) gives

$$\begin{aligned} \partial_{J_{uv}} U_G(I) = (-1)^{k-1} \times 2 \sum_{(p,q)} \sum_{\substack{C \subseteq V \\ a,p \in C,\ q \notin C}} & B_{a,p}(C)\, \frac{Z_H(0)^2}{Z_G(0)^2} \\ & \times\, \partial_{h_{x_2}} \cdots \partial_{h_{x_{2k}}} \left[e^{2[P_H(h) - P_G(h)]}\, \partial_{h_q} P_H(h) \right] \Big|_{h=0}\,. \end{aligned}$$

In each summand, derivatives falling on the magnetization factor produce an Ursell function on $H = G \setminus C$; those falling on the exponential produce products of differences between Ursell functions on H and G. We record this expansion explicitly to make the induction transparent.

Let Π_{2k}^{even} be the set of partitions of the labels $\{1,\dots,2k\}$ into nonempty even-sized blocks, and let D_* denote the block containing label 1. Associate these labels with $(q, x_2, \dots, x_{2k})$, and for any block D we write $U_H(D)$ for the signed Ursell function on H at the vertices indexed by D. Thus the first mark a on the left is replaced by q in the distinguished block D_* on the right. With this notation we may expand:

$$\partial_{J_{uv}} U_G(I) = 2 \sum_{(p,q)} \sum_{\substack{C \subseteq V \\ a,p \in C,\ q \notin C}} B_{a,p}(C)\, \frac{Z_H(0)^2}{Z_G(0)^2} \sum_{\pi \in \Pi_{2k}^{\text{even}}} U_H(D_*) \prod_{\substack{D \in \pi \\ D \neq D_*}} 2\big[U_G(D) - U_H(D)\big]. \tag{A.11}$$

To see the formula, group the derivatives falling on $\partial_{h_q} P_H$ together with its existing q derivative into D_*. The chain rule partitions the remaining derivatives into blocks acting on $2(P_H - P_G)$. Evaluating $h = 0$, the exponential equals 1, and spin-flip symmetry eliminates all odd-sized blocks. To account for the signs: if $|D| = 2k_D$, the distinguished factor contributes a sign $(-1)^{k_{D_*}-1}$, while each difference factor contributes $(-1)^{k_D}$. Their product is $(-1)^{k-1}$, exactly the sign defining $U_G(I)$.

Every term in this recursion is nonnegative. Indeed, H has fewer than N vertices, so the inner induction gives $U_H(D_*) \geq 0$, even when D_* has size $2k$. Every other block D has size at most $2k-2$, since D_* contains at least two labels. For these blocks, the sign rule along with the coupling monotonicity on edges between H and C yields:

$$U_G(D) - U_H(D) \geq 0.$$

Since also $B_{a,p}(C) \geq 0$ and $Z_H(0)^2/Z_G(0)^2 > 0$, the recursion proves $\partial_{J_{uv}} U_G(I) \geq 0$.

This establishes order-$2k$ monotonicity on G for every nonnegative coupling vector J. Increasing the couplings from zero to J now gives

$$U_{G,J}(I) \geq U_{G,0}(I) \geq 0,$$

which proves the sign rule and completes the induction on the number of vertices. When $k = 1$, the only partition consists of the distinguished block, so there are no difference factors and the vertex induction suffices. This starts the induction on k and completes the proof. □

A.1 A counterexample to the restricted-sign claim of Camia, Jiang and Newman

Camia, Jiang and Newman proposed a distinct proof of Ursell-function monotonicity using a combinatorial approach to the random-current expansion of spin correlations [CJN23]. We were unable to verify the proof of the coefficientwise sign assertion in [CJN23, Proposition 1, equation (37)]. In particular, we found a counterexample to the intermediate restricted-sign claim in [CJN23, Proposition 2, equation (43)]. We state that claim as a combinatorial assertion about multigraphs and then give the counterexample, restricting to fourth order to simplify the exposition.

The Setup. Let $M = (V, \mathcal{E})$ be a finite multigraph with distinguished vertices: $u, v \in V$, and let $A = \{j_1, j_2, j_3, j_4\} \subseteq V$ consist of four distinct vertices, with $v \notin A$. The boundary ∂M is the set of vertices of odd degree. We assume $\partial M = A \triangle \{u, v\}$, where $\triangle$ denotes symmetric difference. A partition of M into three labelled sub-multigraphs n_1, n_2, n_3 is an assignment $\mathcal{P} : \mathcal{E} \to \{1, 2, 3\}$. Each subgraph n_i has vertex set V and edge set $\mathcal{P}^{-1}(i)$. Let $\mathcal{C}$ be a set of pairs (e, f) of distinct edge copies. We call $\mathcal{P}$ *restricted* if $\mathcal{P}(e) \neq \mathcal{P}(f)$ for every $(e, f) \in \mathcal{C}$. Among restricted partitions satisfying $u \not\leftrightarrow v$ in $n_1 \cup n_2$ (meaning there is no path from u to v using edges of n_1 and n_2 combined), we define two counts:

1. $P(M; \mathcal{C})$ counts those partitions with
$$\partial n_1 = A \triangle \{u, v\}, \qquad \partial n_2 = \partial n_3 = \varnothing.$$
2. For $Q \subseteq A$ with $|Q| = 2$, $N_Q(M; \mathcal{C})$ counts those partitions with
$$\partial n_1 = Q \triangle \{u, v\}, \qquad \partial n_2 = \varnothing, \qquad \partial n_3 = A \setminus Q.$$

The claim at fourth order. Equation (43) of [CJN23] asserts that, for every such M and every restriction set $\mathcal{C}$ such that no edge appearing in a pair of $\mathcal{C}$ is a self-loop or joins u to v,

$$R(M; \mathcal{C}) := P(M; \mathcal{C}) - \sum_{\substack{Q \subseteq A \\ |Q| = 2}} N_Q(M; \mathcal{C}) \leq 0. \tag{A.12}$$

The counterexample. Take the multigraph $M = (V, \mathcal{E})$ drawn in Figure 1 with seven distinct vertices:

$$V = \{u, v, a, b, j_2, j_3, j_4\}, \qquad \mathcal{E} = \{uj_2, uj_3, ua, ub, av, bv, vj_4\}$$

Take the marked set of vertices $A = \{u, j_2, j_3, j_4\}$ such that $v \notin A$ and $\partial M = \{v, j_2, j_3, j_4\} = A \triangle \{u, v\}$. We impose the restrictions (without loops or edges joining u to v):

$$\mathcal{C} = \{(uj_2, ub),\ (uj_3, ua)\}.$$

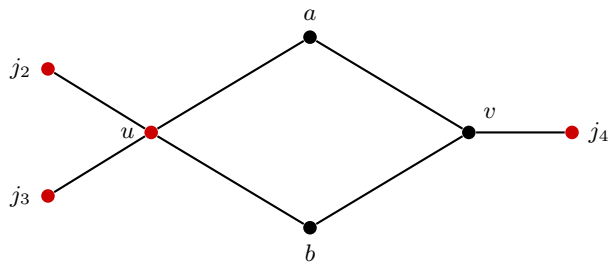


Figure 1: The graph M in the fourth-order counterexample.

We note the following for any admissible partition counted in $P(M;\mathcal{C})$ or $N_Q(M;\mathcal{C})$:

1. Each n_i has even degree at the unmarked degree-two vertices a, b. Thus the two edges of each path $u \to a \to v$ and $u \to b \to v$ must belong to the same subgraph. Since u and v must be disconnected in $n_1 \cup n_2$, both paths must lie in n_3.
2. In all admissible scenarios $\partial n_2 = \emptyset$. Since only leaf edges with a single endpoint remain, this forces n_2 to be empty.
3. The restriction set $\mathcal{C}$ forces uj_2 and uj_3 to belong to different subgraphs from ua and ub, so they cannot belong to n_3. This forces them to belong to n_1.
4. Since v already has degree two in n_3, the edge vj_4 must belong to n_1.

These considerations produce a single admissible partition counted in $P(M;\mathcal{C})$, and no admissible partitions counted in any $N_Q(M;\mathcal{C})$. We conclude that:

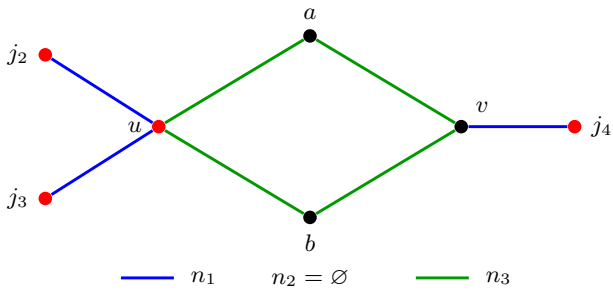


Figure 2: The only admissible partition of M.

$$R(M;\mathcal{C}) = P(M;\mathcal{C}) - \sum_{Q} N_Q(M;\mathcal{C}) = 1 > 0.$$

This contradicts the restricted-sign claim. Without the restrictions, $P(M) = 1$ and precisely three negative contributions remain, one for each of $Q = \{u, j_4\}, \{j_2, j_4\}, \{j_3, j_4\}$, giving $R(M) = -2$. Thus this example does not contradict the unrestricted coefficient inequality.

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