Cluster-Cluster Aggregation on $\mathbb{Z}^2$

Growth and Geometry

TMP
Theoretical and Mathematical Physics

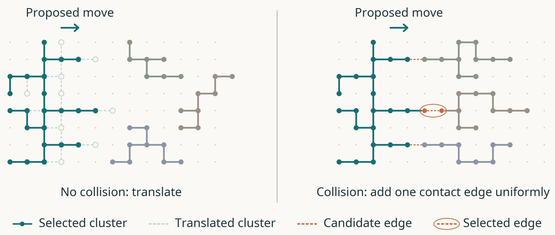
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Master's thesis under Prof. Noam Berger and Dr. Carlo Scali

01 The model

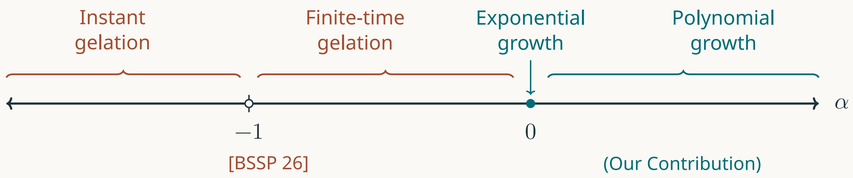
Definition:

- Configurations are subgraphs $\omega \subseteq \mathbb{Z}^d$ with connected components $C \in \mathcal{C}(\omega)$ called **clusters**.
- Initially each site holds a singleton cluster independently with probability p.
- Each cluster $C \in \mathcal{C}(\omega)$ attempts a rigid one-step translation in a uniformly chosen direction with delay $\sim \mathrm{Exp}(|C|^{-\alpha})$. We call α the **mobility exponent**.
- If no collisions occur, the move is carried out. Otherwise the move is suppressed and a uniformly chosen contact bond merges two colliding clusters.
- For $\alpha > -2$, these dynamics give rise to a continuous-time Markov process $\{\omega_t\}_{t\geq 0}$.
- Our main observable is the connected component of the origin $C_0(t)$ and its mass.



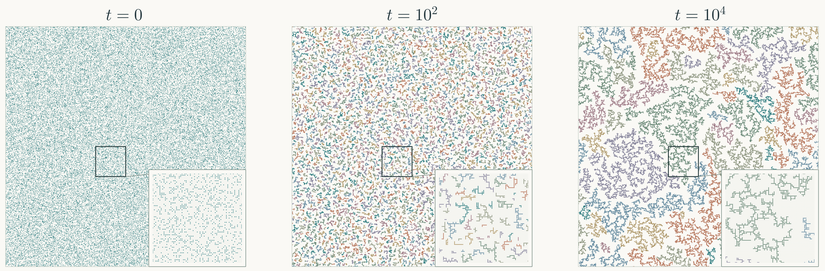
Phase Diagram

- [BSSP 26] proved that an infinite cluster forms in finite time $\iff \alpha < 0$.
- Our work completes the phase diagram by characterizing growth of expected cluster mass at the origin $\mathbb{E}|C_0(t)|$ for $\alpha \geq 0$.



Planar Dynamics

- In this poster we restrict to the planar physical model $\alpha = 1$ and $d = 2$.
- Analogous bounds on $\mathbb{E}|C_0(t)|$ hold for $\alpha > 0$ and $d \geq 2$.
- Clusters steadily grow into fractal dendritic structures. [Meakin 83] estimated their dimension to be between 1.45 and 1.5.



Simulation on a 512×512 periodic lattice, $p = 0.25$, $\alpha = 1$.

02 Growth bounds

Theorem 1: For CCA on $\mathbb{Z}^2$ with mobility exponent $\alpha = 1$ and initial occupancy $p \in (0, 1]$, there exist $c, C > 0$ such that for all $t \geq 0$:

$$c(1+t)^{\frac{1}{2}} \leq \mathbb{E}|C_0(t)| \leq C(1+t)^2$$

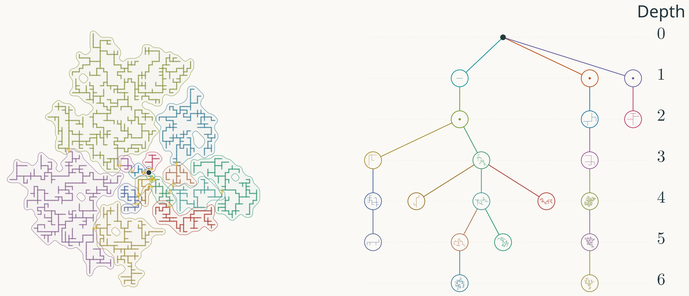
Lower bound

1. Collisions reduce cluster density and force the average mass up.
2. Compare cluster motion to independent random walks up to first collision.
3. Apply heat kernel estimates to lower bound the collision rate.
4. Integrating yields lower bounds for expected cluster size.

Upper bound

1. Clusters grow in layers of successive mergers, indexed by a *merger tree*.
2. In $[-R, R]^2$ the maximum depth of any merger tree is $\mathrm{polylog}(R)$ $\forall\, t \geq 0$.*
3. A cluster with radius R thus requires mergers of radius $O(R)$ to form.
4. These large mergers take at least $O(R)$ time to occur.*
5. Integrating yields a linear upper bound on expected cluster radius in t, hence quadratic bounds on expected mass.

∗ Outside an exceptional event with low probability.



One cluster with highlighted attachments and its corresponding merger tree.

Mean Field Prediction

The Smoluchowski equation describes the mean field evolution of this process. Setting $c_k(t)$ to be the density of clusters of mass k, we have:

$$\frac{dc_k}{dt} = \frac{1}{2}\sum_{i+j=k} K_\alpha(i,j)c_i c_j - c_k \sum_{j\geq 1} K_\alpha(k,j)c_j$$

$$K_\alpha(i,j) = i^{-\alpha}j + j^{-\alpha}i$$

For $\alpha > 0$, this predicts:

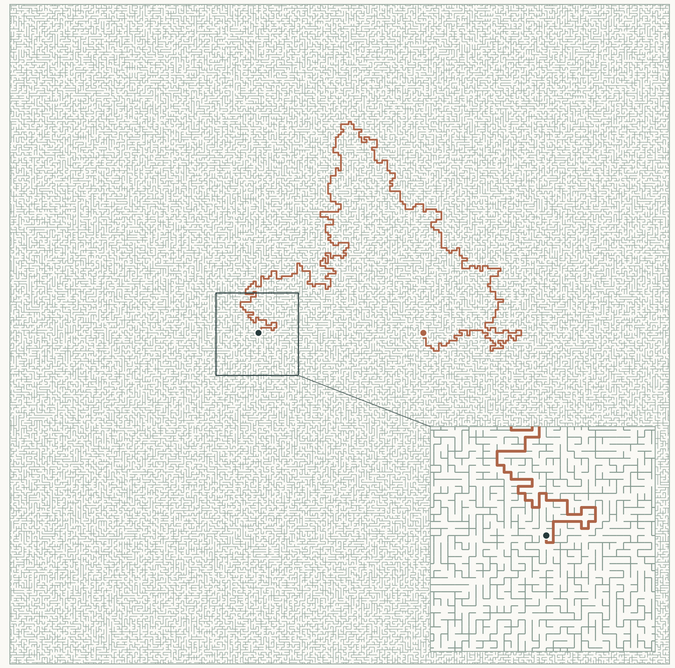
$$\mathbb{E}|C_0(t)| = \sum_{k=1}^{\infty} k^2 c_k(t) \asymp (1+t)^{1/\alpha}.$$

03 Cluster geometry

Theorem 2: For full initial occupancy CCA with $p = 1$, the limiting configuration ω_∞ is a spanning tree of $\mathbb{Z}^2$. In particular, it has a unique connected component.

Proof outline

1. Suppose $\mathbb{P}(0 \not\leftrightarrow e_1 \text{ in } \omega_t) > \varepsilon$ for all $t \geq 0$.
2. Translation invariance implies $|\partial C_0(t)|/|C_0(t)| > \varepsilon$ for all $t > 0$.
3. A positive proportion of this boundary never forms a bond.
4. But the accumulated merger rate of $C_0(t)$ is infinite. ↯



Path $(0,0) \longrightarrow (64,0)$ in ω_∞. Length: 612 edges.

Simulation on a 256×256 periodic lattice, $p = 1$, $\alpha = 1$.

Open problems

- **Sharp growth:** Prove the cluster growth rate predicted by Smoluchowski's equation for $\alpha \geq 0$.
- **Cluster geometry:** How does the graph metric in the limiting configuration scale relative to the ℓ_1 metric? What are the fractal and spectral dimensions of a cluster?

[BSSP 26] N. Berger, E. B. Procaccia, D. Schmid and D. Sharon, *Cluster-Cluster model in* $\mathbb{Z}^d$, arXiv:2608.05105. [Meakin 83] P. Meakin, *Phys. Rev. Lett.* **51**, 1119 (1983). [Smoluchowski, 18] M. v. Smoluchowski, *Z. Phys. Chem.* **92**, 129–168 (1918).